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PARAMETER ESTIMATION USING SEMI FRACTIONAL MOMENTS IN LOGNORMAL DISTRIBUTION

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ABSTRACT. Parameter estimation using method of moment (MOM), maximum likelihood (ML), and method of semi fractional moments (SFM) for lognormal distribution are compared and their corresponding asymptotic covariance matrices are derived. It is found that SFM gives a better estimate than the other two methods.

KEYWORDS. Method of moment, method of maximum likelihood, method of semi fractional moments, lognormal distribution, and asymptotic covariance matrix.

INTRODUCTION

Lognormal distribution has been widely used in many empirical studies in fitting lifetime data models. For example, in the determination of the sizes of organisms, the number of species in the field of biology, the amount of rainfall in meteorology and the sizes of individual incomes in economics. A random variable is normally distributed. The natural logarithm of a normally distributed random variable follows a lognormal distribution.

In this paper we compare results obtained from the three methods; method of moment (MOM), method of maximum likelihood (ML), and method of semi fractional moments (SFM). We estimate the parameters, and derive their asymptotic covariance matrices. From here this paper aims to find the best estimator.

In the conventional MOM lower orders moment are used to estimate the parameters of distributions. It is known that sampling variability of the moment increases as their order is increased. Therefore the order of moments may be reduced to less than 1 with a variable in the interval (0,1) (Masood Rafiq et al 1996), then the r^{th} fractional moment of random variable X with density function $f(x;\theta)$ is defined by:

$$\mu_r = E(x^r) = \int_{-\infty}^{\infty} x^r f(x,\theta)\,dx \tag{1}$$

And the corresponding r^{th} fractional moment from the sample $x_1, x_2, \ldots, x_n$ is defined as:

$$m_r' = \frac{1}{n}\sum_{i=1}^{n} x_i^r \qquad (2)$$

The method of FM will consist of obtaining as many values of the fractional moment as the number of the parameters to be estimated, and equating m_r' with μ_r' for different values of r that is:

$$m_r' = \mu_r' \qquad \text{for} \quad r \in (0,1) \qquad (3)$$

It was found that the method of fractional moment as suggested by (Masood Rafiq et al 1996) gave a good estimate if we take it as a semi fraction moments, by taking one sample moment as in MOM, and the other as fraction moment as suggested in FM. That is better known as Semi fractional moments.

Method of Moment (MOM)

The method of moment determines the estimators of the unknown parameters by equating the sample moments to the corresponding population moments, that is,

$$m_r' = \mu_r' \qquad \text{for} \quad r = 1,2,3,\ldots,k$$

where k is equal to the number of parameters involved: m_r' and μ_r' as defined in (1) and (2) respectively with integer values of r.

Maximum Likelihood Estimate (MLE)

Given a likelihood function $L(\theta)$ for parameter $\theta \in \Theta$, a maximum likelihood estimate of θ is obtained when $L(\hat{\theta})$ is maximum (Adelchi Azzalini 1996).

Semi Fractional Moment Estimate (SFM)

This method determines the estimate of the unknown parameters by equating first the ordinary sample moment by the ordinary population moment and the next fractional sample moment by the fractional population moment. The fractional value is obtained by choosing a suitable distinct values of r that minimizes the determinant of the covariance matrix. It has been observed also that the values of r are invariant for a given value of parameters (see empirical studies). (Noting that when r has an integer value this method becomes the MOM).

In general, the asymptotic covariance matrix of the moment estimators for parameters θ_1 and θ_2 can be evaluated as follows (Tan and Chang 1972),

$$V\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}=\begin{bmatrix}\frac{\partial\theta_1}{\partial m_1'} & \frac{\partial\theta_1}{\partial m_2'}\\ \frac{\partial\theta_2}{\partial m_1'} & \frac{\partial\theta_2}{\partial m_2'}\end{bmatrix}\begin{bmatrix}\operatorname{var}(m_1') & \operatorname{cov}(m_1',m_2')\\ \operatorname{cov}(m_2',m_1') & \operatorname{var}(m_2')\end{bmatrix}\begin{bmatrix}\frac{\partial\theta_1}{\partial m_1'} & \frac{\partial\theta_2}{\partial m_1'}\\ \frac{\partial\theta_1}{\partial m_2'} & \frac{\partial\theta_2}{\partial m_2'}\end{bmatrix}$$

This can be used to evaluate the asymptotic covariance matrix of the fractional moment estimators. (Noting that the asymptotic covariance matrix of the MLE is attained to Cramer-Rao lower bound (Adelchi Azzalini 1996))

ESTIMATION OF THE PARAMETERS

Consider a Lognormal distribution with parameters μ and σ^2. Then the probability density function of random variable X is defined as:

$$g(x)=\begin{cases}\frac{1}{\sqrt{2\pi}\sigma x}e^{\frac{-(\ln x)-\mu)^2}{2\sigma^2}} & x>0\\ 0 & otherwise\end{cases}$$

Then the estimator of parameters μ and σ^2 are $\hat{\mu}$ and $\hat{\sigma}^2$ respectively and their covariance matrix is given below for the three different methods.

(a) Using the method of moment, the following results of derivations are obtained:

$$\hat{\mu}=2\ln(m_1')-\frac{1}{2}\ln(m_2')$$

$$\hat{\sigma}^2=2\ln(m_2')-\frac{1}{2}\ln(m_1')$$

and the corresponding covariance matrix is:

$$V1\begin{pmatrix}\hat{\mu}\\ \hat{\sigma}^2\end{pmatrix}=\frac{1}{n}\begin{bmatrix}4e^{\sigma^2}+\frac{1}{4}e^{4\sigma^2}-2e^{2\sigma^2}-\frac{9}{4} & 3e^{2\sigma^2}-4e^{\sigma^2}-\frac{1}{2}e^{4\sigma^2}+\frac{3}{2}\\ 3e^{2\sigma^2}-4e^{\sigma^2}-\frac{1}{2}e^{4\sigma^2}+\frac{3}{2} & 4e^{\sigma^2}+e^{4\sigma^2}-4e^{2\sigma^2}-1\end{bmatrix}$$

(b) Using the method of maximum likelihood, the estimator of the parameters can be obtained from the likelihood function

$$L(\mu,\sigma^2)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\,\sigma x_i}e^{\frac{-(\ln(x_i)-\mu)^2}{2\sigma^2}}$$

This is equivalent to

$$\ln(L(\mu,\sigma^2))=\frac{-n}{2}\ln(2\pi)-\frac{n}{2}\ln\sigma^2-\sum_{i=1}^{n}\ln(x_i)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln(x_i)-\mu)^2 \qquad (4)$$

Then differentiating (4) with respect to μ and σ^2, and equating the resulting expression to zero we obtain

$$\hat{\mu}=\frac{\sum_{i=1}^{n}\ln(x_i)}{n}$$

and

$$\hat{\sigma}^2=\frac{\sum_{i=1}^{n}(\ln(x_i)-\hat{\mu})^2}{n}$$

Therefore the corresponding covariance matrix is given by

$$V2\begin{pmatrix}\hat{\mu}\\ \hat{\sigma}^2\end{pmatrix}=\begin{bmatrix}\frac{\hat{\sigma}^2}{n} & 0\\ 0 & \frac{2\hat{\sigma}^4}{n}\end{bmatrix}$$

(c) Using the method of Semi fractional moment and extension of MOM the following entities are obtained:

$$\hat{\mu}=\frac{r_2}{r_1(r_2-r_1)}\ln(m'_{r_1})-\frac{r_1}{r_2(r_2-r_1)}\ln(m'_{r_2})$$

and

$$\hat{\sigma}^2=\frac{2}{r_2(r_2-r_1)}\ln(m'_{r_2})-\frac{2}{r_1(r_2-r_1)}\ln(m'_{r_1})$$

The corresponding elements of covariance matrix are as follows:

$$V3(\hat{\mu}) = \left(\frac{r_2}{r_1(r_2 - r_1)}\right)^2 h_{r_1} - \frac{2}{(r_2 - r_1)^2} h_{r_1 2} + \left(\frac{r_1}{r_2(r_2 - r_1)}\right)^2 h_{r_2}$$

$$V3(\hat{\sigma}^2) = \left(\frac{2}{r_1(r_2 - r_1)}\right)^2 h_{r_1} - \frac{8}{r_1 r_2 (r_2 - r_1)^2} h_{r_1 2} + \left(\frac{2}{r_2(r_2 - r_1)}\right)^2 h_{r_2}$$

and

$$COV3(\hat{\mu}, \hat{\sigma}^2) = -2r_2\left(\frac{1}{r_1(r_2 - r_1)}\right)^2 h_{r_1} + \frac{2(r_1 + r_2)}{r_1 r_2 (r_2 - r_1)^2} h_{r_1 2} - 2r_1\left(\frac{2}{r_2(r_2 - r_1)}\right)^2 h_{r_2}$$

where

$$h_{r_1} = \frac{1}{n}(e^{\sigma^2 r_1^2} - 1)$$

$$h_{r_2} = \frac{1}{n}(e^{\sigma^2 r_2^2} - 1)$$

$$h_{r_1 2} = \frac{1}{n}(e^{\sigma^2 r_1 r_2} - 1)$$

Remarks: It turns out that $\hat{\mu}$ is an unbiased estimate of μ for the three cases, and $\hat{\sigma}^2$ is an unbiased estimate of σ^2 for MOM and SFM, while it is the biased estimator for ML. That is to say that $E(\hat{\sigma}^2) = (\frac{n-1}{n})\sigma^2$ in the case of ML, but the bias goes to 0 at the rate $\frac{1}{n}$ as $n \to \infty$. For comparison sake, it is best if all the estimators are unbiased estimators. The bias can be removed by taking the corrected sample variance as shown below:

$$s^2 = \frac{n}{n-1}\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(\ln(x_i) - \bar{x})^2}{n-1}$$

Detailed discussion of the work in this section and the derivations can be found in Tan and Chang (1972) and Crow and Shimizu (1988). Using all the derived entities above, we apply each method of estimation to the following empirical studies.

EMPIRICAL STUDIES

By using Fortran Power Station linked to IMSL library, the comparison of SFM, ML and MOM estimators in terms of their asymptotic variances is evaluated for different given values of the parameters. These results are tabulated in Table 2, noting that the results of asymptotic variance are independent of the location parameter values. In addition, Table 3 gives the relative efficiencies of ML and MOM estimators with respect to the SFM estimators for the estimated parameters. For SFM we need to fix the value of r_1 to be the first ordinary moment and to determine r_2 as fractional value which minimizes the determinant. It turns out that these values are *invariant*. This is clearly shown in Table 1.

Table 1. For given r_1=1, the fractional value of r_2 which minimized the covariance matrix determinant for known value of the scale parameter.

$\hat{\sigma}^2$	r_1	r_2
0.01	1.00	0.001
0.02	1.00	0.001
0.03	1.00	0.001
0.03	1.00	0.001
0.04	1.00	0.001
0.05	1.00	0.001
0.06	1.00	0.001
0.07	1.00	0.001
0.08	1.00	0.001
0.09	1.00	0.001
0.1	1.00	0.001
0.2	1.00	0.001
0.3	1.00	0.001
0.4	1.00	0.001
0.5	1.00	0.001
0.6	1.00	0.001
0.7	1.00	0.001
0.8	1.00	0.001
0.9	1.00	0.001
1.00	1.00	0.001
2.00	1.00	0.001
3.00	1.00	0.001
4.00	1.00	0.001
5.00	1.00	0.001
10.0	1.00	0.001
15.0	1.00	0.001
20.0	1.00	0.001

This entity is used in claiming a better estimator in the following discussion. (For the purpose of this paper we only include for small sample size, n=20). As can be seen from the earlier section, the determinant is independent of the location parameter. The table indicates that the fractional values are invariant.

Table 2: Asymptotic variance of the parameters.

	VAR($\hat{\mu}$)			VAR($\hat{\sigma}^2$)		
$\hat{\sigma}^2$	MOM	ML	SFM	MOM	ML	SFM
0.01	5.000E-4	5.000E-4	5.000E-4	1.030E-5	1.108E-5	9.998E-6
0.02	1.000E-3	1.000E-3	1.000E-3	4.246E-5	4.432E-5	4.000E-5
0.03	1.500E-3	1.500E-3	1.500E-3	9.844E-5	9.972E-5	9.000E-5
0.04	2.002E-3	2.000E-3	2.000E-3	1.802E-4	1.773E-4	1.599E-4
0.05	2.504E-3	2.500E-3	2.500E-3	2.901E-4	2.770E-4	2.500E-4
0.06	3.008E-3	3.000E-3	3.000E-3	4.303E-4	3.989E-4	3.600E-4
0.07	3.512E-3	3.500E-3	3.500E-3	6.033E-4	5.429E-4	4.900E-4
0.08	4.020E-3	4.000E-3	4.000E-3	8.116E-4	7.091E-4	6.400E-4
0.09	4.529E-3	4.500E-3	4.500E-3	1.057E-3	8.975E-4	8.100E-4
0.10	5.041E-3	5.000E-3	5.000E-3	1.344E-3	1.108E-3	1.000E-3
0.20	1.041E-2	1.000E-2	1.000E-2	7.192E-3	4.432E-3	4.000E-3
0.30	1.676E-2	1.500E-2	1.500E-2	2.155E-2	1.000E-2	9.000E-3
0.40	2.522E-2	2.000E-2	2.000E-2	5.090E-2	1.800E-2	1.600E-2
0.50	3.777E-2	2.500E-2	2.500E-2	1.055E-1	2.800E-2	2.500E-2
0.60	5.770E-2	3.000E-3	3.000E-2	2.015E-1	4.000E-2	3.600E-2
0.70	9.028E-2	3.500E-2	3.500E-2	3.639E-1	5.400E-2	4.901E-2
0.80	1.439E-1	4.000E-2	4.000E-2	6.311E-1	7.100E-2	6.401E-2
0.90	2.319E-1	4.500E-2	4.500E-2	1.01619	9.000E-2	8.102E-2
1.00	3.747E-1	5.000E-2	5.000E-2	1.74570	1.110E-1	1.000E-1
2.00	33.1674	0.10000	0.10000	139.556	4.430E-1	4.002E-1
3.00	1997.99	0.15000	0.15000	8061.02	9.970E-1	9.008E-1
4.00	110789.1	0.20000	0.20000	443720.2	1.77285	1.60192
5.00	6062392	0.25000	0.25000	2.425E+7	2.77000	2.50375
10.0	2.94E+15	0.50000	0.50000	1.17E+16	11.0803	10.0300
15.0	1.42E+24	0.75000	0.75000	5.71E+24	24.9307	22.6014
20.0	6.92E+32	1.00000	1.00000	2.77E+33	44.3213	40.2406

The variances of the two estimators derived from each method increase as the value of the estimator increase. As can be seen from the Table 2 as the parameter increases the variance of the estimated parameters by MOM increased significantly.

Table 3 indicates that the maximum likelihood estimate for μ is equally good compared to SFM. While in estimating σ^2, the SFM method gives a superior estimate. The relative efficiency of MOM decrease as the value of the estimate increases which give significant credit to SFM. In order to enhance the above finding, we examine the above claim mentioned through a numerical example in the following section.

Table 3: Relative efficiency of different estimating methods with respect to SFM

	MOM		MLE	
$\hat{\sigma}^2$	eff($\hat{\mu}$)	Eff($\hat{\sigma}^2$)	eff($\hat{\mu}$)	eff($\hat{\sigma}^2$)
0.01	1.00	0.96	1.00	0.90
0.02	1.00	0.94	1.00	0.90
0.03	0.99	0.91	1.00	0.90
0.04	0.99	0.88	1.00	0.90
0.05	0.99	0.86	1.00	0.90
0.06	0.99	0.83	1.00	0.90
0.07	0.99	0.81	1.00	0.90
0.08	0.99	0.78	1.00	0.90
0.09	0.99	0.76	1.00	0.90
0.10	0.99	0.74	1.00	0.90
0.20	0.95	0.55	1.00	0.90
0.30	0.90	0.41	1.00	0.90
0.40	0.80	0.31	1.00	0.90
0.50	0.66	0.23	1.00	0.90
0.60	0.52	0.17	1.00	0.90
0.70	0.39	0.13	1.00	0.90
0.80	0.28	0.10	1.00	0.90
0.90	0.19	0.079	1.00	0.90
1.00	0.13	0.057	1.00	0.90
2.00	0.003	0.003	1.00	0.90
3.00	7.5E-5	1.1E-4	1.00	0.90
4.00	1.8E-6	3.6E-6	1.00	0.90
5.00	~ 0	~ 0	1.00	0.90
10.0	~ 0	~ 0	1.00	0.90
15.0	~ 0	~ 0	1.00	0.90
20.0	~ 0	~ 0	1.00	0.90

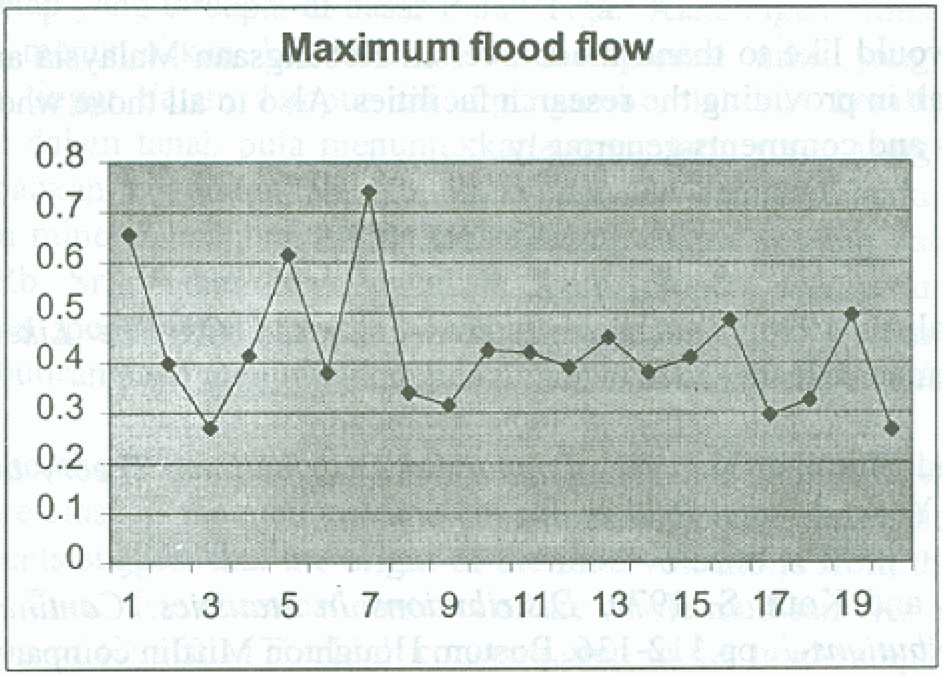
NUMERICAL EXAMPLE

We applied the three methods in determining the parameters concerned by fitting the data to lognormal distribution. We used $r_1 = 1.0$ and $r_2 = 0.001$ to the data published in Crow and Shimizu (1988).

The data reported the maximum flood flow in millions of cubic feet per second for the Susquehanna River at Harrisburg, Pennsylvania, over a 20 four-year intervals from 1890 to 1969 as given by Masood Rafiq, Munir Ahmed and Faqir Muhammed (1996) shown below:

0.654	0.269	0.402	0.416	0.613
0.379	0.740	0.338	0.315	0.423
0.418	0.392	0.449	0.379	0.412
0.484	0.297	0.324	0.494	0.265

These data fit nicely to lognormal distribution as shown by the following figure with mean 0.423125 and variance 0.0156948.



Thus in this work we take r_1 =1.00 and r_2 =0.001 to estimate the two parameters when using SFM method. The results from all the three methods are shown in the following table.

	MOM	ML	SFM
$\hat{\mu}$	- 0.90000	- 0.897700	- 0.897700
$MSE(\hat{\mu})$	0.01570	0.000785	0.000785
$\hat{\sigma}^2$	0.07995	0.076255	0.075291
$MSE(\hat{\sigma}^2)$	5.1671E-4	2.731E-5	2.4779E-5

As can be seen from the above table, the SFM method is more efficient than the other two methods. Noting that the MSE of the MLE variance is computed by using the corrected sample variance as stated in estimation of the parameters.

CONCLUSION AND RECOMMENDATION

As can be seen from the experiments carried out, SFM gives a better estimate than the other two methods for the location, μ and scale parameter, σ^2. The ML gives about

90% efficiency compared to SFM for the scale parameter and is equally efficient for the location parameter. In the case of MOM, the estimates obtained are less efficient compared to SFM (i.e. in the region of 20%) for both the parameters.

Though the procedure is tedious, it is worth making the effort to use SFM method in estimating parameters for non-negative data. Thus thorough work may be carried out further for other distributions especially distributions involving 3 or more parameters.

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