The fixed-points theory revisited with new applications

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Abstract: The fixed-points theory has its root dating back to 1932. Hahnkamm suggested that there are two fixed points in the undamped primary structure's frequency response function (FRF) for the two-degree-of-freedom system when a harmonic force is applied to the primary mass. These points are independent of the damping value in the auxiliary system, which is the control system, and their heights are mainly determined by the mass ratio of the device. The desired optimum value of the tuning ratio is obtained when the heights of the fixed points are equal. Fourteen years later, Brock suggested that the optimum value of the damping ratio in the control device can be determined by making the height of the fixed points the maximum. Since then, the fixed-points theory has been used in many applications as one of the design laws in fabricating a vibration neutralizer. In this article, the theory is reformulated by using the conventional definition of the damping ratio. It is proved that the same result can be obtained as in the original derivation. The application of the theory is then extended to the control of global vibration of an undamped continuous structure and is demonstrated on a simply supported beam.

Keywords: global vibration control, kinetic energy, resonance frequency, vibration neutralizer

1 INTRODUCTION

In the early stage of its application, the vibration neutralizer was normally simply tuned to a particular frequency of interest without any proper selection of its resonance frequency and damping value. However, this situation may excite the two new resonance frequencies of the combined system, making it more problematic when the excitation frequency changes [**1**]. In 1932, Hahnkamm [**2**] realized that for a neutralizer with a given resonance frequency, there are two common points in the frequency response function (FRF) of the primary system regardless of its damping value. Following this, he recommended that the desired value of the neutralizer's resonance frequency is that when the heights of these two common points are equal. However, no remark was made on the damping value in the vibration neutralizer. This leaves the system with two new resonance frequencies.

Fourteen years later, Brock [**3**] devised a mathematical method, known as fixed-points theory, to determine the damping value in the vibration neutralizer that flattens the FRF of the primary structure. This is the optimum value because it does not excite the new resonance frequencies of the system. Since then, the theory has been used as one of the design laws in fabricating a vibration neutralizer [**4**].

In this article, the theory is revisited. First, the theory is discussed using the original approach proposed by Hahnkamm and Brock where the damping in the secondary system was defined in a different way. The theory is then derived using the conventional definition of damping. The application of the theory is then extended to the case of global vibration control of beams.

2 FIXED-POINTS THEORY USING THE ORIGINAL APPROACH

Suppose that an auxiliary spring–mass–damper system is attached to an undamped primary structure. The auxiliary system, widely known as a vibration neutralizer, is used as a control device to dampen the movement of the primary system (Fig. 1). Mathematically, in the original approach of the fixed-points theory, the FRF of the primary system, x_1, is

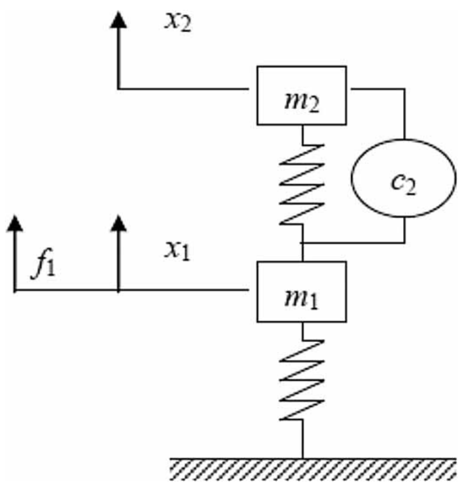

Fig. 1 An auxiliary system, m_2, attached to an undamped problematic primary structure, m_1. The primary system is being forced by a harmonic force f_1

expressed as [2, 3]

$$x_1 = \frac{F_1(k_2 + j\omega c_{o2} - \omega^2 m_2)}{(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - m_2 k_2 \omega^2 + j\omega c_{o2}(k_1 - m_1\omega^2 - m_2\omega^2)} \quad (1)$$

where $F_1, k_1, k_2, m_1, m_2, \omega, j$, and c_{o2} are the amplitude of the excitation force, the spring's stiffness constant of the primary system, the spring's stiffness constant of the vibration neutralizer, mass of the primary system, mass of the vibration neutralizer, frequency of the excitation force, imaginary number, and damping value of the vibration neutralizer (secondary system), respectively. The subscript $_o$ in c_{o2} denotes the usage of the original definition in the fixed-points theory. Here, the neutralizer's damping value is defined as

$$c_{o2} = \zeta_{o2} c_{oc} \quad (2)$$

where ζ_{o2} and c_{oc} are the damping ratio and the critical damping of the vibration neutralizer, respectively. The critical damping is defined as

$$c_{oc} = 2m_2\omega_1 \quad (3)$$

where ω_1 is the circular natural frequency of the primary system given by $\omega_1 = (m_1/k_1)^{1/2}$. Introducing $\mu = m_1/m_2$, the mass ratio; $\omega_2 = (m_2/k_2)^{1/2}$, the undamped resonance frequency of the vibration neutralizer; $f = \omega_2/\omega_1$, the tuning ratio; $x_o = x_{o1}/x_{st}$, the non-dimensional FRF (x_{st} is the static displacement of the primary system); and $g = \omega/\omega_1$, the frequency ratio, equation (1) can be written in terms of the dimensionless parameter as

$$|x_o| = \sqrt{\frac{(2\zeta_{o2}g)^2 + (g^2 - f^2)^2}{\{[(g^2 - 1 + \mu g^2)(2\zeta_{o2}g)]^2 + [\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)]^2\}}} \quad (4)$$

Equation (4) has the form of

$$|x_o| = \sqrt{\frac{(A\zeta_{o2}^2 + B)}{(C\zeta_{o2}^2 + D)}} \quad (5)$$

where

$$A = (2g)^2; \quad B = (g^2 - f^2)^2$$
$$C = [(g^2 - 1 + \mu g^2)(2g)]^2$$
$$D = [\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)]^2 \quad (6)$$

It can be seen that equation (5) satisfies the existence of two invariant points based on the fact that the identity $A/C = B/D$ holds regardless of ζ_{o2} [5].

The two invariant points (or the fixed points) can be shown on a non-dimensional FRF plot of a simple two degree of freedom system as in Fig. 2. In this figure, three FRF plots are shown with different values of ζ_{o2} in comparison with the FRF of the primary structure alone. The mass of the secondary system is 1 per cent of the primary system and its natural frequency is made to coincide with the natural frequency of the primary system. The zero and infinity values of ζ_{o2} were chosen to represent the extreme cases, whereas the 2 per cent ($\zeta_{o2} = 0.02$) represents the intermediate case. It can be seen that the two invariant points P and Q can be proved to be the same for any value of ζ_{o2}.

To determine these invariant points, there are two extreme conditions to be considered: when the damping ratio in the vibration neutralizer is zero and when it is infinity. Mathematically, these two conditions are written as

$$|x_o|_{\zeta_{o2}=0} = \frac{B}{D} \quad (7)$$

$$|x_o|_{\zeta_{o2}=\infty} = \frac{A}{C} \quad (8)$$

The condition $(B/D)^2 = (A/C)^2$ implies the two crossing points of curves $|x_o|_{\zeta_{o2}=0}$ and $|x_o|_{\zeta_{o2}=\infty}$, the fixed points that are being sought. By using this crossing point condition, it can be written that

$$\left[\frac{(g^2 - f^2)}{[\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)]}\right]^2 = \left[\frac{1}{(g^2 - 1 + \mu g^2)}\right]^2 \quad (9)$$

This can be reduced to a simpler form by taking its square roots but a ±ve sign must be added to the right-hand side of the equation. Equation with −ve sign is the trivial solution; therefore the fixed-points equation is given by

$$\frac{(g^2 - f^2)}{[\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)]} = \frac{1}{(g^2 - 1 + \mu g^2)} \quad (10)$$

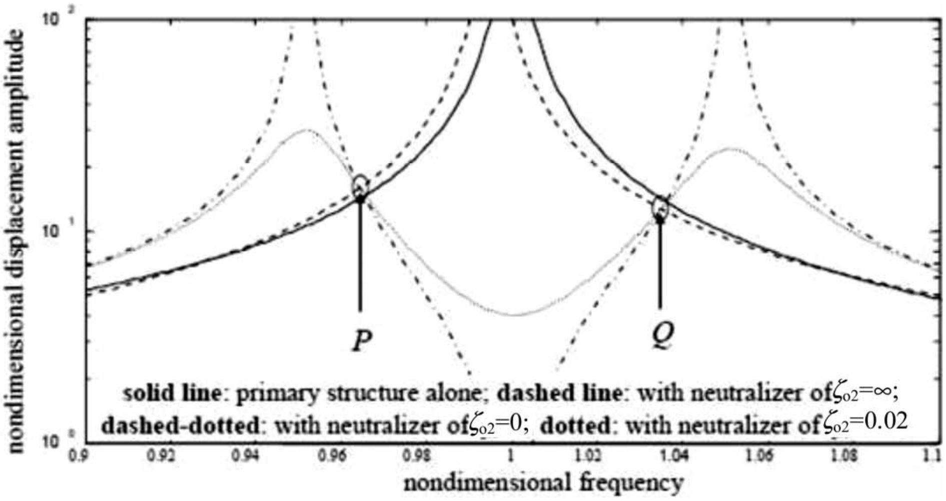

Fig. 2 Non-dimensional FRF of the primary system showing the existence of the two invariant points, P and Q. The tuning ratio is unity ($f = 1$) and $\mu = 0.01$

This leads to

$$g^4 - 2g^2\left(\frac{1+f^2+\mu f}{2+\mu}\right) + f^2\left(\frac{2}{2+\mu}\right) = 0 \tag{11}$$

Equation (11) has two roots, and suppose that g_1^2 and g_2^2 are the roots, then

$$(g^2 - g_1^2)(g^2 - g_2^2) = g^4 - (g_1^2 + g_2^2)g^2 + g_1^2 g_2^2 = 0 \tag{12}$$

Comparing equations (11) and (12), one obtains

$$g_1^2 + g_2^2 = 2\left(\frac{1+f^2+\mu f}{2+\mu}\right) \tag{13}$$

According to the fixed-points theory, the FRF at these two roots must be equal regardless of the damping value in the neutralizer. This occurs when either equation (7) or equation (8) is satisfied. For simplification, equation (8) is used and substituting the two roots yields

$$|x_o|_{\zeta_{o2}=\infty} = \sqrt{\frac{1}{[g_{1,2}(1+\mu)-1]^2}} \tag{14}$$

or

$$|x_o|_{\zeta_{o2}=\infty} = \pm\frac{1}{[g_{1,2}(1+\mu)-1]^2} \tag{15}$$

However, the two equations in equation (15) must be the same according to the fixed-points theory and therefore

$$\frac{1}{[g_{1,2}(1+\mu)-1]} = -\frac{1}{[g_{1,2}(1+\mu)-1]} \tag{16}$$

or

$$g_1^2 + g_2^2 = \frac{2}{2+\mu} \tag{17}$$

Comparing equations (13) and (17), the optimum tuning condition, which is the desirable tuning ratio, is obtained when

$$f_{\mathrm{opt}} = \frac{1}{1+\mu} \tag{18}$$

This optimum condition causes the heights of the two fixed points to be equal. Substituting equation (18) into equation (11) gives the abscissas of the fixed points of the $(g, |x_o|)$ diagram as

$$g_{1,2} = \sqrt{\frac{1 \pm \sqrt{\mu/(2+\mu)}}{1+\mu}} \tag{19}$$

while the common ordinate is

$$|x_o|_{g=g_1} = |x_o|_{g=g_2} = \sqrt{1+\frac{2}{\mu}} \tag{20}$$

which can be derived by substituting equation (19) into equation (15).

The next task is to derive the neutralizer's optimum damping ratio ζ_{o2} that flattens the FRF of the main system. From equation (4), it can be written that

$$\zeta_{o2} = \frac{(g^2-f^2)^2 - x_o^2[\mu f^2 g^2 - (g^2-1)(g^2-f^2)]^2}{4g^2[x_o^2(g^2-1+\mu g^2)^2 - 1]} \tag{21}$$

Suppose that the two fixed points are P and Q. In order that the FRF passes horizontally through the first fixed point P, it is first required that it pass through a point P' of the abscissa

$$g_1 = \sqrt{\frac{1 - \sqrt{\mu/(2+\mu)} + \delta}{1+\mu}} \tag{22}$$

with the ordinate given in equation (20). Then, let δ approach zero as a limit. Substituting equations (18),

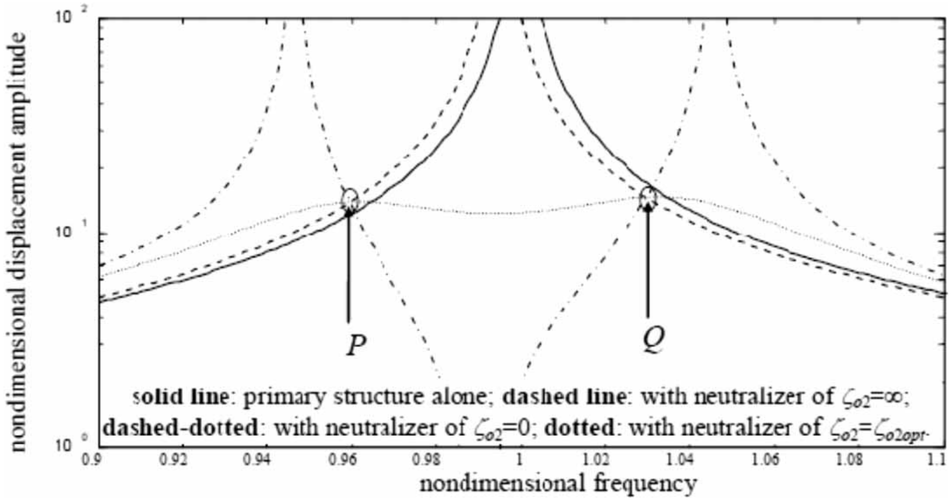

Fig. 3 Non-dimensional FRF of the primary system when the tuning and damping ratios of the neutralizer are optimized, for the original and conventional definitions of damping (dotted line), in comparison with other values (zero and infinity). $\mu = 0.01$

(20), and (22) into equation (21), one will get a result in the form of

$$\zeta_{o2}^2 = \frac{A_o + A_1\delta + A_2\delta^2 + A_3\delta^3 + \cdots}{B_o + B_1\delta + B_2\delta^2 + B_3\delta^3 + \cdots} \tag{23}$$

If $\delta = 0$, the FRF curve lies on the fixed points and ζ_{02} is assumed to be indeterminate because it can take infinity value. Therefore, $A_o = B_o = 0$. If δ is not zero but has a very small value, other terms in equation (23) that were multiplied with δ of power higher than unity can be neglected, leaving only

$$\delta_{o2}^2 = \frac{A_1}{B_1} \tag{24}$$

Therefore, substituting equations (18), (20), and (22) into equation (21) and taking only the terms that are multiplied with δ give after rearrangement

$$\delta_{o2}^2 = \mu\left[\frac{3 - \sqrt{\mu/(2+\mu)}}{8(1+\mu)^3}\right] \tag{25}$$

Following a similar procedure for g_2, one will obtain

$$\zeta_{o2}^2 = \mu\left[\frac{3 + \sqrt{\mu/(2+\mu)}}{8(1+\mu)^3}\right] \tag{26}$$

for a horizontal tangent at point Q. It was suggested in reference [**3**] to take the average of these two damping values as the optimum value, which is given by

$$\zeta_{o2opt} = \sqrt{\frac{3\mu}{8(1+\mu)^3}} \tag{27}$$

With these optimal values of the tuning and damping ratios, the FRF of the system, in terms of displacement, can be made relatively flat. This is shown in Fig. 3 in the non-dimensional FRF plot of the primary system with the secondary system fitted to it, where the mass ratio is kept at 1 per cent, and the tuning and damping ratios are optimized according to equations (18) and (27).

Visual inspection of Fig. 3 shows that when the optimum tuning ratio is used, the heights of the crossing points P and Q are equal, and this is the desired result according to the fixed-points theory. This result cannot be achieved when the natural frequency of the secondary mass is simply made equal to the natural frequency of the primary system as can be seen in Fig. 2.

3 FIXED-POINTS THEORY WITH CONVENTIONAL DAMPING

In the previous section, the fixed-points theory has been discussed in detail. A method to get the optimum tuning and damping ratios of the vibration neutralizer was presented. However, one has to note that the definition of the critical damping in the vibration neutralizer in equation (3) is different from the conventional definition. This is the definition of the critical damping used in the original fixed-points theory.

The conventional definition of the critical damping is given by

$$c_c = 2m_2\omega_2 \tag{28}$$

and therefore the damping ratio of the vibration neutralizer is

$$\zeta_2 = \frac{c}{2m_2\omega_2} \tag{29}$$

Note that the subscript o has been removed as the conventional definition of damping is now being used.

Using this definition, the non-dimensional FRF of the primary system is now written as

$$|x| = \sqrt{\frac{(2\zeta_2 fg)^2 + (g^2 - f^2)^2}{\{[(g^2 - 1 + \mu g^2)(2\zeta_2 fg)]^2 + [\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)]^2\}}} \tag{30}$$

This equation can also be written in the form similar to equation (5) but now with

$$\begin{aligned} A &= (2fg)^2 \\ B &= (g^2 - f^2)^2 \\ C &= [(g^2 - 1 + \mu g^2)(2fg)]^2 \\ D &= [\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)]^2 \end{aligned} \tag{31}$$

It can be seen that all the requirements for the existence of the two invariant points are fulfilled. Therefore, a similar procedure can be followed to obtain the optimum tuning and damping ratios of the neutralizer and this is discussed in the following.

Comparison between Equations (5) and (31) shows that they are identical except in A and C where an additional term f appears in equation (31). However, the additional term cancels out each other in equation (8) and therefore does not alter the procedure in determining the optimal tuning ratio. As a result, the same optimal value for the tuning ratio is obtained as in equation (18). The abscissa and common ordinate also remain the same: respectively

$$g_{1,2} = \sqrt{\frac{1 \pm \sqrt{\mu/(2+\mu)}}{1+\mu}} \tag{32}$$

$$|x|_{g=g_1} = |x|_{g=g_2} = \sqrt{1 + \frac{2}{\mu}} \tag{33}$$

From equation (30), the damping ratio can be written as

$$\zeta_2^2 = \frac{(g^2 - f^2)^2 - x^2[\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)]^2}{4f^2 g^2 [x^2 (g^2 - 1 + \mu g^2)^2 - 1]} \tag{34}$$

By using equation (22) and making δ to approach zero as a limit, it can be written that the damping ratio takes a similar form as in equation (23), which is

$$\zeta_2^2 = \frac{A_o + A_1\delta + A_2\delta^2 + A_3\delta^3 + \cdots}{B_o + B_1\delta + B_2\delta^2 + B_3\delta^3 + \cdots} \tag{35}$$

Using the same argument as before, this equation simplifies to

$$\zeta_2^2 = \frac{A_1}{B_1} \tag{36}$$

Substituting equations (18), (32), and (33) into equation (34) leads to

$$\zeta_2^2 = \mu\left[\frac{3 - \sqrt{\mu/(2+\mu)}}{8(1+\mu)}\right] \tag{37}$$

and

$$\zeta_2^2 = \mu\left[\frac{3 + \sqrt{\mu/(2+\mu)}}{8(1+\mu)}\right] \tag{38}$$

respectively. Taking the average of these two values gives the optimum value of the neutralizer's damping ratio as

$$\zeta_{2\text{opt}} = \sqrt{\frac{3\mu}{8(1+\mu)}} \tag{39}$$

The optimum value of the tuning ratio obtained from the two different methods was proved to be the same. However, the optimum value of the damping ratio was found to be different. This is because of the difference in definition of the damping in the vibration neutralizer. Even then, the desired FRF of the system remains the same as shown in Fig. 3. It can be seen that the optimized FRF using the conventional definition overlaps the FRF using the original definition of the neutralizer's damping terms indicating similar performance.

4 APPLICATION OF THE FIXED-POINTS THEORY TO GLOBAL VIBRATION CONTROL

The kinetic energy of a continuous structure, as a measure of the global vibration of the structure, is written as [**6**]

$$KE = \frac{M_s\omega^2}{4}(\boldsymbol{q}^{\mathrm{H}}\boldsymbol{q}) \tag{40}$$

where M_s, ω, and $\boldsymbol{q}$ are the mass of the structure, the circular frequency of the excitation force, and the vector of the modal amplitude, respectively. The superscript H denotes the Hermitian transpose. For a structure with a well-separated natural frequency, the kinetic energy in the vicinity of the natural frequency can be well approximated by [**7**]

$$KE_m = \frac{M_s\omega^2}{4}(q_m^* q_m) \tag{41}$$

where q_m is the mth mode of the modal amplitude given by

$$q_m = \frac{A_m g_m}{1 + K_k A_m \varphi_m^2(x_k)} \tag{42}$$

Here A_m, g_{pm}, K_k, $\varphi_m(x_k)$, and x_k are the mth mode of the complex modal amplitude, the generalized primary force in its mth mode, the neutralizer's dynamic stiffness, the mth mode of the mode shape of the structure, and the neutralizer's location on the structure, respectively. The superscript $*$ denotes the complex conjugate.

To simplify the problem, a simply supported beam is selected as a primary structure, and for an undamped beam, the complex modal amplitude is

$$A_m = \frac{1}{k_m - M_s\omega^2} \tag{43}$$

while the dynamic stiffness of the neutralizer is

$$K_k = -M_k\omega^2 \left(\frac{k_k + j2\zeta_k\sqrt{M_k K_k}\omega}{k_k - M_k\omega^2 + j2\zeta_k\sqrt{M_k K_k}\omega} \right) \tag{44}$$

respectively. k_k, M_k, and ζ_k are the stiffness constant, mass and damping ratio of the neutralizer. k_m in equation (43) is the mth effective bending stiffness of the beam given by

$$k_m = (m\pi)^4 \left(\frac{El}{L^3} \right) \tag{45}$$

where E, I, and L are the Young's modulus, moment inertia, and the length of the beam, respectively. Note that the conventional definition of the damping ratio of the neutralizer is used, which is given by

$$\zeta_k = \frac{c}{2m_k\omega_k} \tag{46}$$

where ω_k is the neutralizer's resonance frequency given by $\omega_k = (m_k/k_k)^{1/2}$. Therefore, equation (41) becomes after rearrangement

$$q_m = \frac{g_{\mathrm{pm}}[(j2\zeta_k\sqrt{M_k k_k}\omega) + (k_k - M_k\omega^2)]}{\begin{array}{c}\{(j2\zeta_k\sqrt{M_k k_k}\omega)(k_m - M_s\omega^2 - M_k\omega^2\varphi_m^2(x_k)) \\ + [(k_m - M_s\omega^2)(k_k - M_k\omega^2) \\ - M_k K_k\omega^2\varphi_m^2(x_k)]\}\end{array}} \tag{47}$$

Taking

$$\mu = \frac{M_k}{M_s}, \quad f_m = \frac{\omega_k}{\omega_m}, \quad \text{and} \quad g_m = \frac{\omega}{\omega_m} \tag{48}$$

and substituting equation (47) into equation (41) give after rearrangement

$$\gamma_m = \frac{(A\zeta_k^2 + B)}{(C\zeta_k^2 + D)} \tag{49}$$

where

$$\begin{aligned} \gamma_m &= KE_m \left(\frac{4m^4\pi^4 EI}{M_s L^3\omega^2 g_{\mathrm{pm}}^2} \right) \\ A &= (2f_m g_m)^2 \\ B &= (g_m^2 - f_m^2)^2 \\ C &= [(g_m^2 - 1 + \mu g_m^2\varphi_m^2(x_k))(2f_m g_m)]^2 \\ D &= [\mu f_m^2 g_m^2\varphi_m^2(x_k) - (g_m^2 - 1)(g_m^2 - f_m^2)]^2 \end{aligned} \tag{50}$$

This is the non-dimensional kinetic energy of the beam. It can be seen that equation (49) has a similar form as the non-dimensional displacement in equation (31) for the simple primary system. Therefore, all the requirements for the existence of fixed points are fulfilled. This implies that a similar procedure can be used to determine the optimum tuning and damping ratios of the neutralizer that flatten the non-dimensional kinetic energy of the beam. By following a similar procedure, the optimal tuning ratio of the neutralizer is found to be

$$f_{m\mathrm{opt}} = \frac{1}{1 + \mu\varphi_m^2(x_k)} \tag{51}$$

The abscissa and common ordinate are respectively

$$g_{m_{1,2}} = \sqrt{\frac{1 \pm \sqrt{[\mu\varphi_m^2(x_k)]/[2 + \mu\varphi_m^2(x_k)]}}{1 + \mu\varphi_m^2(x_k)}} \tag{52}$$

$$\gamma_m|_{g=g_1} = \gamma_m|_{g=g_2} = 1 + \frac{2}{\mu\varphi_m^2(x_k)} \tag{53}$$

The damping ratio in equation (49) can be expressed as

$$\zeta_k^2 = \frac{\begin{array}{c}(g_m^2 - f_m^2)^2 - \gamma_m[\mu\varphi_m^2(x_k)f_m^2 g_m^2 \\ - (g_m^2 - 1)(g_m^2 - f_m^2)]\end{array}}{4f_m^2 g_m^2[\gamma_m(g_m^2 - 1 + \mu\varphi g_m^2(x_k)g_m^2) - 1]} \tag{54}$$

Again, suppose that the two fixed points on the non-dimensional kinetic energy curve are P and Q. In order that the curve passes horizontally through the first fixed point P, it is required that it pass through a point P' of the abscissa

$$g_{m_1} = \sqrt{\frac{1 - \sqrt{[\mu\varphi_m^2(x_k)]/[2 + \mu\varphi_m^2(x_k)]} + \delta}{1 + \mu\varphi_m^2(x_k)}} \tag{55}$$

with the ordinate given in equation (53). Then, let δ approach zero as a limit. Substituting equations (51), (53), and (55) into equation (54) gives a result in the form of

$$\zeta_k^2 = \frac{A_o + A_1\delta + A_2\delta^2 + A_3\delta^3 + \cdots}{B_o + B_1\delta + B_2\delta^2 + B_3\delta^3 + \cdots} \tag{56}$$

The same procedure as in the previous section can be followed, which gives the damping ratio as

$$\zeta_k^2 = \mu\varphi_m^2(x_k) \left[\sqrt{\frac{3 - \sqrt{[\mu\varphi_m^2(x_k)]/[2 + \mu\varphi_m^2(x_k)]}}{8[1 + \mu\varphi_m^2(x_k)]}} \right] \tag{57}$$

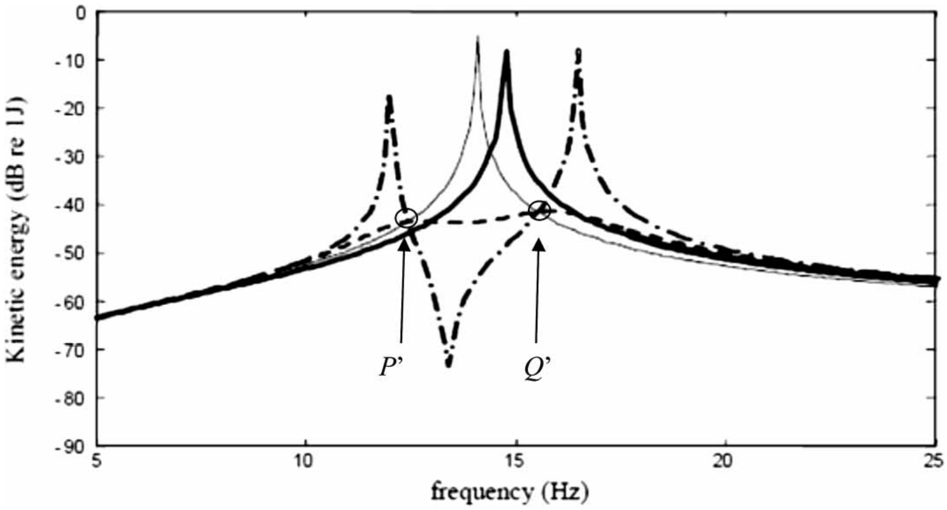

Fig. 4 Effects on the kinetic energy of a beam with the application of the optimized vibration neutralizer. Solid bold – no control; solid – $\zeta_k = \infty$; dashed dotted – $\zeta_k = 0$; dashed line – $\zeta_k =$ optimum. $x_k = 0.5L$, and the control target is the first natural frequency of the beam (15 Hz)

Similarly, the non-dimensional kinetic energy curve passes horizontally through the point Q when the damping ratio is given by

$$\zeta_k^2 = \mu\varphi_m^2(x_k)\left[\sqrt{\frac{3+\sqrt{[\mu\varphi_m^2(x_k)]/[2+\mu\varphi_m^2(x_k)]}}{8[1+\mu\varphi_m^2(x_k)]}}\right] \tag{58}$$

In turn, the optimal damping ratio of the neutralizer can be found by taking the average between equations (57) and (58), which is

$$\zeta_{k\mathrm{opt}} = \left[\frac{3\mu\varphi_m^2(x_k)}{8(1+\mu\varphi_m^2(x_k))}\right] \tag{59}$$

To visualize the performance of the optimized tuning and damping ratios of a vibration neutralizer on global vibration of a structure, a numerical simulation of the kinetic energy of a simply supported beam with an optimized neutralizer fitted to it was carried out. The beam has the following properties: physical dimensions of 1 m × 0.0381 m × 0.006 35 m; material density 7870 kg/m^3 and Young's modulus 207 GN/m^2, and unity amplitude of a primary point force is applied at $0.1L$ ($x_\mathrm{f} = 0.1L$) with a neutralizer fitted at $x_k = 0.5L$. Figure 4 shows the total kinetic energy of the beam contributed from the first ten modes in its first natural frequency. The solid boldface line represents the uncontrolled beam. As in the previous section, it can be seen that invariant points (P' and Q') exist that are the crossing points of the beam's kinetic energy of all values of the neutralizer's damping ratio. The interesting fact here is that the global behaviour of the beam, which is the kinetic energy, is almost flat when the tuning and damping ratios of the neutralizer are optimized. This is shown by the dashed line in the figure. This shows that the fixed-points theory can also be used to flatten the global response of a continuous structure such as beams.

5 CONCLUSION

In this article, the original fixed-points theory has been first reviewed. The original theory uses a different definition of the damping value in the vibration neutralizer. It was found that using a similar procedure, the same value of the neutralizer's optimal tuning ratio can be obtained even if the conventional definition of the neutralizer's damping value is used. However, the value of the optimal damping ratio is found to be different. Nevertheless, they result in similar performance. The theory was then used to determine the optimum tuning and damping ratios of the neutralizer for the case of global vibration control of a simply supported beam. It was shown that the theory can also be used to flatten the global response of a continuous structure such as beams.

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APPENDIX

Notation

A_m	the mth mode of the complex modal amplitude
c_c	critical damping of the vibration neutralizer in the conventional definition
c_{oc}	critical damping of the vibration neutralizer in its original definition
c_{o2}	damping value of the vibration neutralizer in its original definition
c_2	damping value of the vibration neutralizer in the conventional definition
f	tuning ratio
f_{opt}	optimum tuning frequency
F_1	amplitude of the excitation force
g	frequency ratio
g_{pm}	the generalized primary force in its mth mode acting on the host structure
H	Hermitian transpose
I	moment inertia of the beam
j	imaginary number
k_k	stiffness constant of the vibration neutralizer on a continuous structure
k_m	the mth effective bending stiffness of the beam on a continuous structure
k_1	the spring's stiffness constant of the primary system
k_2	the spring's stiffness constant of the secondary system
K_k	the neutralizer's dynamic stiffness on a continuous structure
KE	kinetic energy of the host structure
L	length of the beam
m_1	mass of the primary system
m_2	mass of the secondary system
M_k	mass of the vibration neutralizer on a continuous structure
M_s	mass of the host structure
$\boldsymbol{q}$	vector of the modal amplitude
q_m	the mth mode of the modal amplitude of the host structure
x_f	location of the excitation force on the host structure
x_k	the neutralizer's location on the host structure
x_o	non-dimensional frequency response function
x_{o1}	displacement amplitude of the primary system
x_{st}	static displacement of the primary system
x_1	FRF of the primary system
Y	Young's modulus of the beam
γ_m	non-dimensional kinetic energy of the beam
ζ_k	damping ratio of the neutralizer on a continuous structure
ζ_{o2}	damping of the vibration neutralizer in its original definition
ζ_{o2opt}	optimum damping of the vibration neutralizer in its original definition
ζ_2	damping of the vibration neutralizer in the conventional definition
ζ_{2opt}	optimum damping of the vibration neutralizer in the conventional definition
μ	mass ratio
φ_m	the mth mode of the mode shape of the host structure
ω	circular frequency of the excitation force
ω_k	the neutralizer's resonance frequency on a continuous structure
ω_1	circular natural frequency of the primary system
ω_2	undamped resonance frequency of the vibration neutralizer